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Novel Classes of Sets in Ideal Čech Closure Spaces

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Abstract

Relaxing the idempotent condition in closure operations leads to a generalized framework for topological spaces, offering significant advantages in modeling proximity. This work explores the extension of classical closure concepts within the framework of ideal Čech closure spaces. This extension establishes the idea of generalized Čech closed sets relative to an ideal $\mathbb{L}$, known as $\mathbb{L}\underset{\cdot}{g}$-Čclosed sets. This definition aims to expand the theoretical and practical utility of closure notions by incorporating ideals, allowing for more comprehensive, and applicable frameworks in mathematical scopes. Additionally, the research investigates the properties of these $\mathbb{L}\underset{\cdot}{g}$-Čclosed sets, highlighting their role in offering new insights into topological structures and closure operations in settings where traditional closure conditions are insufficient. Also, we have developed an algorithm for identifying $\mathbb{L}\underset{\cdot}{g}$-Čclosed sets, which provides a figurative representation to enhance understanding of the underlying concepts and computational steps. These novel ideas lead to the introduction of new kinds, namely $\Omega^{\mathbb{L}}$-sets and $\mho^{\mathbb{L}}$-sets, within Čech closure spaces. Specifically, the structure $(\Upsilon, \Omega^{\mathbb{L}})$ forms a convex closure space, and several fundamental properties related to these sets are established. Furthermore, by providing counterexamples to one-sided theorems, the research demonstrates situations where the converse does not hold, thus spotlighting the complexities and limitations of these generalized frameworks.

Keywords: Čech closure space; ideals; $\mathbb{L}\underset{\cdot}{g}$-Čclosed and $\mathbb{L}\underset{\cdot}{g}$-Čopen sets; $\Omega^{\mathbb{L}}$-sets and $\mho^{\mathbb{L}}$-sets.

1 Introduction

Different generalized structures, including weak structures [11], generalized topologies [10], Čech closure spaces [9], and others, have been extensively studied. These structures have played a significant role in extending classical topological concepts and addressing complex applications across diverse disciplines. In global applications–such as medical diagnosis and decision-making problems related to COVID-19 [16], as well as economic applications [14]–topological structures have demonstrated notable effectiveness.

Among these, Čech closure spaces have lately gained significant attention from researchers due to their applications in a wide range of areas, such as medicine [4], image processing [5], networks [17], digital topology [28], information systems [24] and homotopy theory [8, 26]. These applications illustrate the flexibility and practicality of the Čech closure framework in solving problems where classical topological approaches may prove insufficient. Čech closure spaces, initially presented in [9], serve as a generalization of Kuratowski closure spaces [22]. The distinguishing feature of Čech closure spaces is the exclusion of the idempotent condition, which enlarges their applicability and enables the modeling of structures and behaviors not captured by traditional closure operations. This relaxation constitutes Čech closure spaces especially valuable in applied fields that demand adjustable mathematical contexts.

In classical topology, the idea of closed sets is foundational for understanding the structure of spaces through closure operators. Levine [23] pioneered the concept of generalized closed sets (g̣-closed sets) within topological spaces, thereby extending the idea of closed sets. This extension has been further investigated in the context of biweak structure spaces [1], Čech closure spaces [12], generalized topological spaces [21], and soft topology [18]. Ideals [22], which are pivotal constructs in general topology, have played a crucial role in expanding the fundamental properties of topology. The foundational contributions of Jankovic and Hamlett [20] established the basis for employing ideals to extend fundamental topological properties. This approach has been further developed and applied in subsequent works [27]. The study of specific classes of g̣-closed sets defined by ideals has led to the discovery of several new properties [32]. Jafari and Rajesh [19] extended the notion of g̣-closed sets by introducing the idea of g̣-closed sets relative to an ideal, referred to as $\mathbb{L}$g̣-closed sets.

This paper contributes to the discussion by introducing and analyzing g̣-closed sets relative to ideals within a Čech closure space ($\mathbb{L}$g̣-Čclosed sets), as an improvement of the original concepts of g̣-closed sets in Čech closure space (g̣-Čclosed sets) as explored in [7], as well as $\mathbb{L}$g̣-closed sets in topological spaces of Jafari and Rajesh [19]. Fundamental operations such as unions, intersections, and subspaces concerning $\mathbb{L}$g̣-Čclosed sets within this context are investigated. The paper also introduces the concept of $\mathbb{L}$g̣-Čopen sets, offering a comprehensive examination of their properties. Also, an algorithm for identifying $\mathbb{L}$g̣-Čclosed is presented. Moreover, by using $\mathbb{L}$g̣-Čopen and $\mathbb{L}$g̣-Čclosed sets, new sorts of sets that called $\Omega^{\mathbb{L}}$-sets and $\mho^{\mathbb{L}}$-sets respectively, in Čech closure spaces are introduced. Besides, it has been demonstrated that $(\Upsilon, \Omega^{\mathbb{L}})$ constitutes a convex closure space, as described by Stadler [29]. Furthermore, the paper demonstrates that many results from earlier studies can be regarded as special cases of the findings presented herein.

The primary objective of this research is offering a unifying framework that integrates multiple closure operations, including those found in topology, and generalized topology. Specifically, the research seeks to achieve the following goals:

- Extending the notion of closed sets beyond traditional closure spaces by incorporating ideal structures, allowing for a broader class of sets that can capture more comprehensive or subtlety topological properties.
- Providing greater flexibility in how closure properties are described and handled.
- Establishing new notions and results within the framework of ideal Čech closure spaces.
- Exploring new properties and relationships of sets in Čech closure spaces enhanced with ideals. This may include the investigation of continuity in the context of $\mathbb{L}$g̣-Čclosed sets.
- Improving the structural aspects of topological spaces where ideal-based closure operations can make more accurate insights compared to classical approach.
- Developing tools for applications in domains such as topology, rough set theory, and data analysis where conventional notions of closed sets may not be enough.

The motivation behind introducing $\mathbb{L}$g̣-Čclosed sets in ideal Čech closure spaces stems from the need to:

- Generalize classical closed sets concepts as explored in [7, 19] and offer more flexible and adaptable structures, broadening their applicability.
- Incorporate ideals into Čech closure spaces, these generalized sets allow for a more refined and controlled way of dealing with subsets.
- Enhance descriptions of topological properties like continuity, that provides new insights into how topological structures behave when $\mathbb{L}$g̣-Čclosed sets are applied.

2 Preliminaries

Similar to topological spaces, Čech closure spaces comprise a nonempty set Υ equipped with a closure operator $\check{c}l$ defined by specific axioms. This general framework accommodates a broader range of closure operations while retaining fundamental properties of classical closure concepts.

Definition 2.1. [9] *Let* $\check{c}l : 2^{\Upsilon} \to 2^{\Upsilon}$, *where* 2^{Υ} *denotes the power set of* Υ. *The function* $\check{c}l$ *is referred to as a closure operator if it satisfies the next axioms:*

1. *Empty set condition:* $\check{c}l(\emptyset) = \emptyset$.
2. *Extensivity (enlarging property):* $M \subseteq \check{c}l(M)\ \forall M \subseteq \Upsilon$.
3. *Additivity (finite union property):* $\check{c}l(\psi \cup M) = \check{c}l(\psi) \cup \check{c}l(M)\ \forall \psi, M \subseteq \Upsilon$.
4. *Idempotency:* $\check{c}l(M) = \check{c}l(\check{c}l(M))\ \forall M \subseteq \Upsilon$.

When the operator $\check{c}l$ *satisfies Axioms 1, 2 and 3, it is defined as a Čech closure operator, and* $(\Upsilon, \check{c}l)$ *is referred to as a Čech closure space. A topological space may be regarded a specific instance of a Čech closure space, where the operator* $\check{c}l$ *additionally satisfies the idempotent property (Axiom 4). In this context, the Čech closure space* $(\Upsilon, \check{c}l)$ *is simply referred to as a closure space.*

In a Čech closure space $(\Upsilon, \check{c}l)$, a subset M is named Čech closed (denoted as Čclosed) if $\check{c}l(M) = M$. A subset is defined as Čopen if its complement is Čclosed. The families of all Čclosed sets and Čopen sets in Υ are denoted by $\breve{C}C(\Upsilon)$ and $\breve{C}O(\Upsilon)$, respectively. It is evident that both $\emptyset$ and Υ are simultaneously Čopen and Čclosed in $(\Upsilon, \check{c}l)$.

Definition 2.2. [9] *The subsets ψ, M of a Čech closure space $(\Upsilon, \check{c}l)$ are considered separated if they satisfy the following conditions:*

$$\check{c}l(\psi) \cap M = \emptyset, \quad \textit{and} \quad \psi \cap \check{c}l(M) = \emptyset.$$

Definition 2.3. [9] *Let $(\Upsilon, \check{c}l)$ denote a Čech closure space. The associated Čech interior operation for Υ, denoted by $\hat{i}nt_{\check{c}} : 2^{\Upsilon} \to 2^{\Upsilon}$, is given by:*

$$\hat{i}nt_{\check{c}}(M) = \Upsilon \setminus \check{c}l(\Upsilon \setminus M).$$

The Čech interior operation possesses the following properties:

- $\hat{i}nt_{\check{c}}(\Upsilon) = \Upsilon.$
- $\hat{i}nt_{\check{c}}(M) \subseteq M \; \forall M \subseteq \Upsilon.$
- $\hat{i}nt_{\check{c}}(\psi \cap M) = \hat{i}nt_{\check{c}}(\psi) \cap \hat{i}nt_{\check{c}}(M) \; \forall \psi, M \subseteq \Upsilon.$

A subset M is named Čech open if $\check{i}nt_{\check{c}}(M) = M$.

Definition 2.4. [9] *A Čech closure space $(Y, \check{c}l_Y)$ is considered a subspace of $(\Upsilon, \check{c}l)$ if $Y \subseteq \Upsilon$ and the closure operator $\check{c}l_Y(\psi)$ is defined by:*

$$\check{c}l_Y(\psi) = \check{c}l(\psi) \cap Y, \quad \textit{for every} \quad \psi \subseteq Y.$$

If Y is Čclosed in $(\Upsilon, \check{c}l)$, then the subspace $(Y, \check{c}l_Y)$ is also referred to as Čech closed. Furthermore, if E is a Čclosed subset of $(Y, \check{c}l_Y)$, and $(Y, \check{c}l_Y)$ is a Čech closed subspace of $(\Upsilon, \check{c}l)$, then E is a Čclosed set relative to $(\Upsilon, \check{c}l)$.

Definition 2.5. [9] *A function $\dot{f} : (\Upsilon, \check{c}l) \to (\Lambda, \check{c}l^{\star})$ between two Čech closure spaces $(\Upsilon, \check{c}l)$, $(\Lambda, \check{c}l^{\star})$ is named Čech-continuous if,*

$$\dot{f}(\check{c}l(\psi)) \subseteq \check{c}l^{\star}(\dot{f}(\psi)), \; \forall \; \psi \subseteq \Upsilon.$$

Equivalently, $\dot{f}$ is Čech-continuous if,

$$\check{c}l(\dot{f}^{-1}(M)) \subseteq \dot{f}^{-1}(\check{c}l^{\star}(M)), \; \forall \; M \subseteq \Lambda.$$

Another characterization of Čech-continuity is that $\dot{f}^{-1}(H)$ is a Čclosed set relative to $(\Upsilon, \check{c}l)$ whenever H is a Čclosed set relative to $(\Lambda, \check{c}l^{\star})$.

Definition 2.6. [9] *A function $\dot{f} : (\Upsilon, \check{c}l) \to (\Lambda, \check{c}l^{\star})$ is referred to as:*

1. *Čech-closed if $\dot{f}(H)$ is a Čclosed set relative to $(\Lambda, \check{c}l^{\star})$ whenever H is a Čclosed set relative to $(\Upsilon, \check{c}l)$.*
2. *Čech-open if $\dot{f}(H)$ is a Čopen set relative to $(\Lambda, \check{c}l^{\star})$ whenever H is a Čopen set relative to $(\Upsilon, \check{c}l)$.*

Definition 2.7. [7] *Consider $(\Upsilon, \check{c}l)$ being a Čech closure space. A set $\psi \subseteq \Upsilon$ is named a generalized Čech closed set (abbreviated as $\dot{g}$-Čclosed), if $\check{c}l(\psi) \subseteq G$ when G is a Čopen set relative to $(\Upsilon, \check{c}l)$ with $\psi \subseteq G$. A set ψ is termed a generalized Čech open set (abbreviated as $\dot{g}$-Čopen) if its complement is a $\dot{g}$-Čclosed set. The class of all $\dot{g}$-Čclosed (resp. $\dot{g}$-Čopen) sets is specified as $G\breve{C}C(\Upsilon)$ (resp. $G\breve{C}O(\Upsilon)$).*

Proposition 2.1. [7] *Let $(\Upsilon, \check{c}l)$ denote a Čech closure space. The next features have been established:*

1. *A Čclosed set is g̣-Čclosed.*
2. *Union of two g̣-Čclosed sets is g̣-Čclosed.*
3. *If M is a Čclosed, ψ is a g̣-Čclosed, then $M \cap \psi$ is g̣-Čclosed.*
4. *A subset ψ is Čclosed, if it is both Čopen and g̣-Čclosed.*

Definition 2.8. [20, 22] *An ideal $\mathbb{L}$ on Υ is a nonempty class of subsets of Υ that satisfies the next terms:*

1. *Heredity: $\psi \in \mathbb{L}$, $M \subseteq \psi \Rightarrow M \in \mathbb{L}$, $\forall\ \psi, M \subseteq \Upsilon$,*
2. *Finite additivity: $\psi \in \mathbb{L}$, $M \in \mathbb{L} \Rightarrow \psi \cup M \in \mathbb{L}$, $\forall\ \psi, M \subseteq \Upsilon$.*

Henceforth, let $(\Upsilon, \check{c}l, \mathbb{L})$ denote a nonempty set Υ equipped with a Čech closure operator $\check{c}l$ and an ideal $\mathbb{L}$.

3 On $\mathbb{L}$g̣-Čclosed Sets

The inclusion of ideals offers a more flexible framework for analyzing g̣-closed sets in Čech closure spaces and exploring their practical applications across various domains. This part is devoted to present the notions of g̣-Čech closed and g̣-Čech open sets relative to an ideal $\mathbb{L}$. Additionally, we examine some key properties of these sets.

Definition 3.1. *Consider $(\Upsilon, \check{c}l, \mathbb{L})$ being an ideal Čech closure space. A set ψ of Υ is named a generalized Čech closed set relative to an ideal $\mathbb{L}$ ($\mathbb{L}$g̣-Čclosed), if $\check{c}l(\psi) \setminus G \in \mathbb{L}$ when G is a Čopen set and $\psi \subseteq G$.*

Note: For a detailed explanation of the computation process for generalized Čech closed sets relative to an ideal $\mathbb{L}$ (referred to as $\mathbb{L}$g̣-Čclosed sets), please refer to Figure 1, which ensures a visual representation to improve understanding of the primary notions and computational steps.

Definition 3.2. *A set ψ of Υ is named generalized Čech open set relative to an ideal $\mathbb{L}$ ($\mathbb{L}$g̣-Čopen), if $(\Upsilon \setminus \psi)$ is $\mathbb{L}$g̣-Čclosed. The family of all $\mathbb{L}$g̣-Čopen (resp. $\mathbb{L}$g̣-Čclosed) sets of an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$ is indicated as $\mathbb{L}\d{g}\check{C}O(\Upsilon)$ (resp. $\mathbb{L}\d{g}\check{C}C(\Upsilon)$).*

Theorem 3.1. *A set ψ is $\mathbb{L}$g̣-Čopen in an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$ iff $F \setminus M \subseteq \check{int}(\psi)$ for a certain $M \in \mathbb{L}$, when $F \subseteq \psi$ and F is Čclosed set.*

Proof. ($\Rightarrow$) Presume that ψ is an $\mathbb{L}$g̣-Čopen set. Let F be a Čclosed subset of ψ. Then, we have

$$(\Upsilon \setminus \psi) \subseteq (\Upsilon \setminus F).$$

By the assumption that ψ is $\mathbb{L}$g̣-Čopen, it follows that there exists some $M \in \mathbb{L}$ such that,

$$\check{c}l(\Upsilon \setminus \psi) \subseteq (\Upsilon \setminus F) \cup M.$$

Input : γ (nonempty set), $\mathbb{I}$ (ideal), $\hat{\mathbf{c}}\mathbf{l}: 2^{\gamma} \to 2^{\gamma}$ (Ĉech closure operator), $\mathbf{\Psi}$ (subset of γ).
.**Output :** " $\mathbf{\Psi}$ is a generalized Ĉech closed set with respect to $\mathbb{I}$ " or " $\mathbf{\Psi}$ is not a generalized Ĉech closed set with respect to an ideal $\mathbb{I}$ "

Steps :

1. Initialize:

- Let $\mathbf{\Psi} \subseteq \gamma$ be the subset under consideration.
- Let $\mathbb{I}$ be the ideal on γ.
- Let $\hat{C}O(\gamma)$ be the collection of all Ĉopen sets G such that $\mathbf{\Psi} \subseteq$ G.

2. For Each Ĉopen Set G in $\hat{C}O(\gamma)$:

- Compute $\hat{\mathbf{c}}\mathbf{l}(\mathbf{\Psi})$.
- Calculate $\hat{\mathbf{c}}\mathbf{l}(\mathbf{\Psi}) \setminus$G.

3. Check Ideal Membership:

- Verify whether $\hat{\mathbf{c}}\mathbf{l}(\mathbf{\Psi}) \setminus G \in \mathbb{I}$.

4. Decision Criteria:

- If for all Ĉopen sets G where $\mathbf{\Psi} \subseteq$ G, the condition $\hat{\mathbf{c}}\mathbf{l}(\mathbf{\Psi}) \setminus G \in \mathbb{I}$ holds, then:
  - **Output:** $\mathbf{\Psi}$ is a generalized Čech closed set with respect to $\mathbb{I}$.
- Otherwise:
  - **Output:** $\mathbf{\Psi}$ is **not** a generalized Čech closed set with respect to $\mathbb{I}$.

Figure 1: Algorithm for identifying generalized Čech closed sets relative to an ideal $\mathbb{L}$ ($\mathbb{L}$g-Čclosed sets).

Rearranging, we obtain:

$$F \setminus M \subseteq \check{int}(\psi).$$

($\Leftarrow$) Presume that for every Čclosed set F contained in ψ, there exists some $M \in \mathbb{L}$ such that,

$$F \setminus M \subseteq \check{int}(\psi).$$

Consider any Čopen set U such that,

$$(\Upsilon \setminus \psi) \subseteq U.$$

Then,

$$(\Upsilon \setminus U) \subseteq \psi.$$

As per the assumption, there exists $M \in \mathbb{L}$ such that,

$$(\Upsilon \setminus U) \setminus M \subseteq \check{int}(\psi) = \Upsilon \setminus \check{c}l(\Upsilon \setminus \psi).$$

This gives that,

$$\Upsilon \setminus (U \cup M) \subseteq \Upsilon \setminus \check{c}l(\Upsilon \setminus \psi),$$

which implies,

$$\check{c}l(\Upsilon \setminus \psi) \subseteq (U \cup M).$$

Since $M \in \mathbb{L}$, it follows that $\breve{c}l(\Upsilon \setminus \psi)$ is contained in U, $U \in \mathbb{L}$. Consequently,

$$\breve{c}l(\Upsilon \setminus \psi) \in \mathbb{L},$$

which means that $\Upsilon \setminus \psi$ is $\mathbb{L}\d{g}$-Čclosed. Hence, ψ is $\mathbb{L}\d{g}$-Čopen. □

Remark 3.1. *In an ideal Čech closure space* $(\Upsilon, \breve{c}l, \mathbb{L})$*:*

1. *The empty set* $\emptyset$*, and the universal set* Υ *are both* $\mathbb{L}\d{g}$*-Čclosed and* $\mathbb{L}\d{g}$*-Čopen.*
2. *If* ψ *is Čopen and* $\mathbb{L}\d{g}$*-Čclosed set, then its symmetric difference with its Čech closure* $\breve{c}l(\psi)$ *belongs to the ideal* $\mathbb{L}$ *i.e.* $\psi \bigtriangleup \breve{c}l(\psi) \in \mathbb{L}$.

Remark 3.2.

1. *Definition 3.1 in this article provides a generalization of Definition 3.1 from [7], implying that* $G\breve{C}O(\Upsilon) \subseteq \mathbb{L}\d{g}\breve{C}O(\Upsilon)$.
2. *When* $\mathbb{L} = \{\emptyset\}$*, the two concepts coincide, meaning that* $G\breve{C}O(\Upsilon) = \mathbb{L}\d{g}\breve{C}O(\Upsilon)$.
3. *If the universal set* Υ *contains only a single element, such as* $\Upsilon = \{a\}$*, then* $G\breve{C}O(\Upsilon) = \mathbb{L}\d{g}\breve{C}O(\Upsilon)$ *for any ideal* $\mathbb{L}$.

Example 3.1. *Consider the set* $\Upsilon = \{1, 2, 3\}$ *equipped with the following Čech closure operator:*

$$\breve{c}l(\emptyset) = \emptyset, \qquad \breve{c}l(\{1\}) = \{1, 2\},$$
$$\breve{c}l(\{2\}) = \breve{c}l(\{3\}) = \breve{c}l(\{2, 3\}) = \{2, 3\},$$
$$\breve{c}l(\{1, 2\}) = \breve{c}l(\{1, 3\}) = \breve{c}l(\Upsilon) = \Upsilon.$$

Suppose that $\mathbb{L} = \{\emptyset, \{2\}\}$ *is an ideal on* Υ*. Then, the class of all various types of open sets are determined as follows:*

- *The family of Čopen sets,* $\breve{C}O(\Upsilon)$*, is given by* $\breve{C}O(\Upsilon) = \{\emptyset, \Upsilon, \{1\}\}$.
- *The family of generalized Čopen sets,* $G\breve{C}O(\Upsilon)$*, is given by* $G\breve{C}O(\Upsilon) = \{\emptyset, \Upsilon, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}$.
- *The family of* $\mathbb{L}\d{g}$*-Čopen sets,* $\mathbb{L}\d{g}\breve{C}O(\Upsilon)$*, consists of all subsets of* Υ*, meaning* $\mathbb{L}\d{g}\breve{C}O(\Upsilon) = \mathcal{P}(\Upsilon)$.

Proposition 3.1. *Let* $\mathbb{L}, \mathcal{L}$ *be ideals on a Čech closure space* $(\Upsilon, \breve{c}l)$*. If* $\mathbb{L} \subseteq \mathcal{L}$*, then every* $\mathbb{L}\d{g}$*-Čclosed set is* $\mathcal{L}\d{g}$*-Čclosed.*

Proof. By applying Definition 3.1, the outcome follows immediately. □

Remark 3.3. *Let* Υ *be a set containing more than one element, equipped with the Čech closure operator defined as follows:*

$$\breve{c}l(\emptyset) = \emptyset,\ \breve{c}l(M) = \Upsilon, \quad \text{for a nonempty subset } M \text{ of } \Upsilon.$$

Under this closure operator, it follows that for any ideal $\mathbb{L}$ *on* Υ*, every subset of* Υ *is* $\mathbb{L}\d{g}$*-Čclosed, that is*

$$P(\Upsilon) = \mathbb{L}\d{g}\breve{C}C(\Upsilon)).$$

In the next part, we will begin to study some properties of $\mathbb{L}$g̣-Čclosed sets.

Proposition 3.2. *Consider $(\Upsilon, \check{c}l, \mathbb{L})$ as an ideal Čech closure space. Hence, each g̣-Čclosed set is also $\mathbb{L}$g̣-Čclosed.*

Proof. Let $\psi \subseteq G$, where G is a Čopen set. Since ψ is g̣-Čclosed, it follows that $\check{c}l(\psi) \subseteq G$. Hence, we obtain $\check{c}l(\psi) \setminus G = \emptyset$, which belongs to $\mathbb{L}$ as $\emptyset \in \mathbb{L}$. Therefore, ψ is $\mathbb{L}$g̣-Čclosed set. □

In general, the converse of Proposition 3.2 may not hold. The next example illustrates this point and supports our claim.

Example 3.2. *Continuation of Example 3.1.*

Consider the ideal $\mathbb{L} = \{\emptyset, \{2\}\}$ on Υ. In this case, the set $\{1\}$ is $\mathbb{L}$g̣-Čclosed, however, it does not satisfy the conditions to be g̣-Čclosed.

Proposition 3.3. *Let $\mathbb{L}$ be an ideal on a Čech closure space $(\Upsilon, \check{c}l)$. A set ψ of Υ is considered $\mathbb{L}$g̣-Čclosed if, for every $q \in \check{c}l(\psi)$, the condition $\check{c}l(\{q\}) \cap \psi \notin \mathbb{L}$ holds.*

Proof. Let G be a Čopen set with $\psi \subseteq G$. Suppose that $\check{c}l(\psi) \setminus G \notin \mathbb{L}$, then $\check{c}l(\psi) \setminus G \neq \emptyset$. Consequently, a point q exists such that $q \in \check{c}l(\psi), q \in (\Upsilon \setminus G)$. Since $\Upsilon \setminus G$ is a Čclosed set containing q, then $\check{c}l(\{q\}) \subseteq \check{c}l(\Upsilon \setminus G) = \Upsilon \setminus G$ and so $\check{c}l(\{q\}) \cap G = \emptyset \in \mathbb{L}$. Since $\psi \subseteq G$, then $\check{c}l(\{q\}) \cap \psi \in \mathbb{L}$ i.e $\check{c}l(\{q\}) \cap \psi \in \mathbb{L}$, for any $q \in \check{c}l(\psi)$, that is a contradiction. Consequently, $\check{c}l(\psi) \setminus G \in \mathbb{L}$ i.e ψ is $\mathbb{L}$g̣-Čclosed. □

The subsequent example demonstrates that the converse of Proposition 3.3 is not valid.

Example 3.3. *Continued from Example 3.1.*

Let $\mathbb{L} = \{\emptyset, \{2\}\}$ be an ideal on Υ. Consider $\psi = \{2\}$. It can be observed that ψ is $\mathbb{L}$g̣-Čclosed, while the condition $\check{c}l(\{q\}) \cap \psi \in \mathbb{L}$ holds for any $q \in \check{c}l(\psi)$.

Proposition 3.4. *Consider $\mathbb{L}$ as an ideal on a Čech closure space $(\Upsilon, \check{c}l)$. Then, for each $q \in \Upsilon$, one of the following holds:*

1. *$\{q\}$ is Čclosed, or*
2. *$\Upsilon \setminus \{q\}$ is $\mathbb{L}$g̣-Čclosed.*

Proof. Presume that $\{q\}$ is not a Čclosed set. Then, $\Upsilon \setminus \{q\}$ is not Čopen. Let G be any Čopen set with $\Upsilon \setminus \{q\} \subseteq G$. Hence, $G = \Upsilon$. Thus, we have

$$\check{c}l(\Upsilon \setminus \{q\}) \setminus G = \emptyset \in \mathbb{L}.$$

Therefore, $\Upsilon \setminus \{q\}$ is an $\mathbb{L}$g̣-Čclosed set. □

Theorem 3.2. *Let ψ, M be $\mathbb{L}$g̣-Čclosed sets in an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$. Their union, $\psi \cup M$, is likewise $\mathbb{L}$g̣-Čclosed.*

Proof. Let G be a Čopen set with $\psi \cup M \subseteq G$. Then, we have $\psi \subseteq G$ and $M \subseteq G$. Since ψ, M are both $\mathbb{L}$g̣-Čclosed sets, it follows that,

$$\check{c}l(\psi) \setminus G \in \mathbb{L}, \quad \text{and} \quad \check{c}l(M) \setminus G \in \mathbb{L}.$$

Hence, we conclude that,

$$\check{c}l(\psi \cup M) \setminus G \in \mathbb{L}.$$

In other words, $\psi \cup M$ is $\mathbb{L}$g̣-Čclosed. □

Remark 3.4. *The intersection of two $\mathbb{L}$g̣-Čclosed sets is not guaranteed to be $\mathbb{L}$g̣-Čclosed, as evidenced by the next example.*

Example 3.4. *Continuing from Example 3.1.*

Suppose that $\mathbb{L} = \{\emptyset, \{1\}\}$ is an ideal on Υ. In this case, the sets $\{1, 2\}$ and $\{1, 3\}$ are $\mathbb{L}$g̣-Čclosed sets, but their intersection, $\{1\}$, is not $\mathbb{L}$g̣-Čclosed.

Corollary 3.1. *Consider $\mathbb{L}$ being an ideal on a Čech closure space $(\Upsilon, \check{c}l)$. If ψ, M are $\mathbb{L}$g̣-Čopen sets, then $\psi \cap M$ is $\mathbb{L}$g̣-Čopen.*

Proof. Let ψ, M be $\mathbb{L}$g̣-Čopen sets. Since $\Upsilon \setminus \psi, \Upsilon \setminus M$ are $\mathbb{L}$g̣-Čclosed, by Theorem 3.2, we conclude that $\Upsilon \setminus (\psi \cap M)$ is $\mathbb{L}$g̣-Čclosed. Consequently, this implies that $\psi \cap M$ is $\mathbb{L}$g̣-Čopen. □

Theorem 3.3. *Let ψ be an $\mathbb{L}$g̣-Čclosed set in an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$. Then, $\psi \cap F$ is an $\mathbb{L}$g̣-Čclosed, if F is a Čclosed set.*

Proof. Presume that U is a Čopen set with $\psi \cap F \subseteq U$. Then, we have $\psi \subseteq U \cup (\Upsilon \setminus F)$. Since ψ is $\mathbb{L}$g̣-Čclosed and $U \cup (\Upsilon \setminus F)$ is a Čopen set, then,

$$\check{c}l(\psi) \setminus [U \cup (\Upsilon \setminus F)] \in \mathbb{L}.$$

Since F is a Čclosed set, we have $\check{c}l(\psi \cap F) \subseteq \check{c}l(\psi) \cap F = (\check{c}l(\psi) \cap F) \setminus (\Upsilon \setminus F)$. As a result,

$$\check{c}l(\psi \cap F) \setminus U \subseteq (\check{c}l(\psi) \cap F) \setminus (U \cup (\Upsilon \setminus F)) \subseteq \check{c}l(\psi) \setminus [U \cup (\Upsilon \setminus F)] \in \mathbb{L}.$$

Hence, $\psi \cap F$ is $\mathbb{L}$g̣-Čclosed. □

Theorem 3.4. *Let ψ be an $\mathbb{L}$g̣-Čclosed in an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$. If $F \subseteq \check{c}l(\psi) \setminus \psi$ and F is a Čclosed set, then $F \in \mathbb{L}$.*

Proof. Presume that F is a Čclosed subset of $\check{c}l(\psi) \setminus \psi$, meaning that $F \subseteq \check{c}l(\psi)$ and $F \subseteq (\Upsilon \setminus \psi)$. Hence, we have $\psi \subseteq \Upsilon \setminus F$. Since ψ is $\mathbb{L}$g̣-Čclosed and $\Upsilon \setminus F$ is Čopen set, it follows that $\check{c}l(\psi) \setminus (\Upsilon \setminus F) \in \mathbb{L}$. This implies that $\check{c}l(\psi) \cap F \in \mathbb{L}$. Since $F \subseteq \check{c}l(\psi)$, we conclude that $F \in \mathbb{L}$. □

Remark 3.5. *The next example demonstrates that the converse of Theorem 3.4 does not hold in a Čech closure space. However, in closure spaces, both the theorem and its converse are valid, as demonstrated in Theorem 1 [19].*

Example 3.5. *Consider the ideal* $\mathbb{L} = \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$ *on* $\Upsilon = \{a,b,c,d\}$. *Define a Čech closure operator* $\breve{c}l$ *on* Υ *as:*

$$\breve{c}l(\emptyset) = \emptyset, \quad \breve{c}l(\{a\}) = \{a\}, \quad \breve{c}l(\{b\}) = \breve{c}l(\{a,b\}) = \{a,b\}, \quad \breve{c}l(\{c\}) = \breve{c}l(\{c,d\}) = \{b,c,d\},$$
$$\breve{c}l(\{d\}) = \{d\}, \qquad \breve{c}l(\{a,d\}) = \{a,d\}, \qquad \breve{c}l(\{b,d\}) = \breve{c}l(\{a,b,d\}) = \{a,b,d\},$$
$$\breve{c}l(\{a,c\}) = \breve{c}l(\{b,c\}) = \breve{c}l(\{a,b,c\}) = \breve{c}l(\{a,c,d\}) = \breve{c}l(\{b,c,d\}) = \breve{c}l(\Upsilon) = \Upsilon.$$

Let $\psi = \{b,c\}$. *It is clear that for any Čclosed* F *s.t.* $F \subseteq \breve{c}l(\psi) \setminus \psi$, *then* $F \in \mathbb{L}$, *However,* ψ *itself is not* $\mathbb{L}\d{g}$*-Čclosed set.*

Corollary 3.2. *Consider* $(\Upsilon, \breve{c}l, \mathbb{L})$ *as an ideal Čech closure space. Then, the set* $\breve{c}l(\psi) \setminus \psi$ *is a Čclosed that belongs to the ideal* $\mathbb{L}$, *if* ψ *is a Čclosed set.*

Proof. Let ψ be a Čclosed set. Then, we have $\breve{c}l(\psi) \setminus \psi = \emptyset$. Since the empty set $\emptyset$ is always Čclosed, it follows that $\breve{c}l(\psi) \setminus \psi$ is a Čclosed set. Moreover, as $\emptyset \in \mathbb{L}$, we conclude that $\breve{c}l(\psi) \setminus \psi \in \mathbb{L}$. □

Theorem 3.5. *Let* $(Y, \breve{c}l_Y, \mathbb{L}_Y)$ *be a Čech closed subspace of* $(\Upsilon, \breve{c}l, \mathbb{L})$, *where* $\mathbb{L}$ *and* $\mathbb{L}_Y = \{F \cap Y : F \in \mathbb{L}\}$ *are ideals defined on* Υ, Y, *respectively. Let* $\psi \subseteq Y \subseteq \Upsilon$. *If* ψ *is* $\mathbb{L}_Y\d{g}$*-Čclosed in* $(Y, \breve{c}l_Y, \mathbb{L}_Y)$, *then it is* $\mathbb{L}\d{g}$*-Čclosed in* $(\Upsilon, \breve{c}l, \mathbb{L})$.

Proof. Let U be a Čopen set in $(\Upsilon, \breve{c}l, \mathbb{L})$ and $\psi \subseteq U$. Then, we have $\psi \subseteq U \cap Y$. Since ψ is $\mathbb{L}_Y$ẹg-Čclosed and $(U \cap Y)$ is a $\breve{C}_Y$open set, it follows that $\breve{c}l_Y(\psi) \setminus (U \cap Y) \in \mathbb{L}_Y$. Now, observe that,

$$[\breve{c}l(\psi) \setminus U] \cap Y = [\breve{c}l(\psi) \cap Y] \setminus (U \cap Y) = \breve{c}l_Y(\psi) \setminus (U \cap Y) \in \mathbb{L}_Y.$$

Consequently, $\breve{c}l(\psi) \setminus U \in \mathbb{L}$, and so ψ is $\mathbb{L}\d{g}$-Čclosed in Υ. □

Theorem 3.6. *Let* $\dot{f} : (\Upsilon, \breve{c}l, \mathbb{L}) \to (\Lambda, \breve{c}l^\star, \dot{f}(\mathbb{L}))$ *be a Čech continuous Čech closed function, where* $\dot{f}(\mathbb{L}) = \{\dot{f}(M) : M \in \mathbb{L}\}$. *If* ψ *is* $\mathbb{L}\d{g}$*-Čclosed in* Υ, *then* $\dot{f}(\psi)$ *is* $\dot{f}(\mathbb{L})\d{g}$*-Čclosed in* Λ.

Proof. Consider $U \subseteq \Lambda$ being a Čopen set with $\dot{f}(\psi) \subseteq U$. Then, we have $\psi \subseteq \dot{f}^{-1}(U)$. Since $\dot{f}$ is a Čech continuous function, then $\dot{f}^{-1}(U) \subseteq \Upsilon$ is a Čopen. Since ψ is $\mathbb{L}\d{g}$-Čclosed in Υ, it follows that $\breve{c}l(\psi) \setminus \dot{f}^{-1}(U) \in \mathbb{L}$. Therefore, $\dot{f}[\breve{c}l(\psi) \setminus \dot{f}^{-1}(U)] \in \dot{f}(\mathbb{L})$. Next, we observe that,

$$\dot{f}(\breve{c}l(\psi)) \setminus U \subseteq \dot{f}[\breve{c}l(\psi) \setminus \dot{f}^{-1}(U)],$$

so $\dot{f}(\breve{c}l(\psi)) \setminus U \in \dot{f}(\mathbb{L})$. Since $\dot{f}$ is Čech closed function, we have

$$\breve{c}l^\star(\dot{f}(\psi)) \subseteq \breve{c}l^\star(\dot{f}(\breve{c}l(\psi))) = \dot{f}(\breve{c}l(\psi)).$$

Thus, we have

$$\breve{c}l^\star(\dot{f}(\psi)) \setminus U \subseteq \dot{f}(\breve{c}l(\psi)) \setminus U \in \dot{f}(\mathbb{L}).$$

Consequently, $\dot{f}(\psi)$ is $\dot{f}(\mathbb{L})\d{g}$-Čclosed in Λ. □

The following theorems and corollaries will be realized only in the presence of an idempotent condition.

Theorem 3.7. *Consider $\mathbb{L}$ as an ideal on an idempotent Čech closure space $(\Upsilon, \breve{cl})$. If ψ is an $\mathbb{L}\dot{g}$-Čclosed set and $\psi \subseteq G \subseteq \breve{cl}(\psi)$, then G is $\mathbb{L}\dot{g}$-Čclosed.*

Proof. Suppose that H is Čopen and $G \subseteq H$. Since $\psi \subseteq G$, it follows that $\psi \subseteq H$. Given that ψ is an $\mathbb{L}\dot{g}$-Čclosed set, we have $\breve{cl}(\psi) \setminus H \in \mathbb{L}$. As $\breve{cl}$ is idempotent operator, and given that $G \subseteq \breve{cl}(\psi)$, it follows that $\breve{cl}(G) \subseteq \breve{cl}(\psi)$. Consequently, $\breve{cl}(G) \setminus H \in \mathbb{L}$, implying that G is $\mathbb{L}\dot{g}$-Čclosed. □

Corollary 3.3. *Consider $\mathbb{L}$ as an ideal on an idempotent Čech closure space $(\Upsilon, \breve{cl})$. If ψ is an $\mathbb{L}\dot{g}$-Čclosed set, then $\breve{cl}(\psi)$ is also $\mathbb{L}\dot{g}$-Čclosed.*

Theorem 3.8. *Consider $\mathbb{L}$ being an ideal on an idempotent Čech closure space $(\Upsilon, \breve{cl})$. A set ψ is $\mathbb{L}\dot{g}$-Čclosed iff $\breve{cl}(\psi) \setminus \psi$ is $\mathbb{L}\dot{g}$-Čopen.*

Proof. $(\Rightarrow)$ Presume that F is Čclosed such that $F \subseteq \breve{cl}(\psi) \setminus \psi$. Since ψ is an $\mathbb{L}\dot{g}$-Čclosed set, regarding to Theorem 3.4, it follows that $F \in \mathbb{L}$. Hence, $F \setminus U = \emptyset$ for a certain $U \in \mathbb{L}$. Obviously, $F \setminus U \subseteq \breve{int}(\breve{cl}(\psi) \setminus \psi)$. According to Theorem 3.1 $\breve{cl}(\psi) \setminus \psi$ is $\mathbb{L}\dot{g}$-Čopen.

$(\Leftarrow)$ Presume that G is a Čopen set with $\psi \subseteq G$. Then,

$$\breve{cl}(\psi) \cap (\Upsilon \setminus G) \subseteq \breve{cl}(\psi) \cap (\Upsilon \setminus \psi) = \breve{cl}(\psi) \setminus \psi.$$

Since $\breve{cl}(\psi) \cap (\Upsilon \setminus G)$ is a Čclosed, by the hypothesis, we have

$$[\breve{cl}(\psi) \cap (\Upsilon \setminus G)] \setminus U \subseteq \breve{int}(\breve{cl}(\psi) \setminus \psi) = \emptyset,$$

for a certain $U \in \mathbb{L}$. This implies that,

$$\breve{cl}(\psi) \cap (\Upsilon \setminus G) \subseteq U \in \mathbb{L}.$$

Therefore, $\breve{cl}(\psi) \setminus G \in \mathbb{L}$, and thus ψ is $\mathbb{L}\dot{g}$-Čclosed. □

Theorem 3.9. *Consider $\mathbb{L}$ as an ideal on an idempotent Čech closure space $(\Upsilon, \breve{cl})$. If ψ, M are separated $\mathbb{L}\dot{g}$-Čopen sets, then the union $\psi \cup M$ is $\mathbb{L}\dot{g}$-Čopen.*

Proof. Assume that ψ, M are separated $\mathbb{L}\dot{g}$-Čopen sets. Let F be a Čclosed subset of $\psi \cup M$. Then,

$$F \cap \breve{cl}(\psi) \subseteq \psi, \quad \text{and} \quad F \cap \breve{cl}(M) \subseteq M.$$

From the idempotency of $\breve{cl}$ operator, both $F \cap \breve{cl}(\psi)$, $F \cap \breve{cl}(M)$ are Č closed sets. From supposition, there exist $U, V \in \mathbb{L}$ such that,

$$(F \cap \breve{cl}(\psi)) \setminus U \subseteq \breve{int}(\psi), \quad \text{and} \quad (F \cap \breve{cl}(M)) \setminus V \subseteq \breve{int}(M).$$

This implies that,

$$[(F \cap \breve{cl}(\psi)) \setminus \breve{int}(\psi)] \subseteq U \in \mathbb{L}, \quad \text{and} \quad [(F \cap \breve{cl}(M)) \setminus \breve{int}(M)] \subseteq V \in \mathbb{L}.$$

Therefore, we have

$$[(F \cap \breve{cl}(\psi)) \setminus \breve{int}(\psi)] \cup [(F \cap \breve{cl}(M)) \setminus \breve{int}(M)] \in \mathbb{L}.$$

Hence,

$$[F \cap (\check{c}l(\psi) \cup \check{c}l(M)) \setminus (\check{i}nt(\psi) \cup \check{i}nt(M))] \in \mathbb{L}.$$

Since $F = F \cap (\psi \cup M) \subseteq F \cap \check{c}l(\psi \cup M)$, we deduce that,

$$F \setminus \check{i}nt(\psi \cup M) \subseteq [F \cap \check{c}l(\psi \cup M)] \setminus \check{i}nt(\psi \cup M) \subseteq [F \cap \check{c}l(\psi \cup M)] \setminus [\check{i}nt(\psi) \cup \check{i}nt(M)] \in \mathbb{L}.$$

Consequently, $F \setminus U \subseteq \check{i}nt(\psi \cup M)$, for a certain $U \in \mathbb{L}$. This concludes that $\psi \cup M$ is $\mathbb{L}$g̣-Čopen. □

Corollary 3.4. *Consider $\mathbb{L}$ as an ideal on an idempotent Čech closure space $(\Upsilon, \check{c}l)$. Let ψ, M be $\mathbb{L}$g̣-Čclosed sets, then the intersection $\psi \cap M$ is $\mathbb{L}$g̣-Čclosed, provided that $\Upsilon \setminus \psi, \Upsilon \setminus M$ are separated.*

Based on Remark 3.1 and Corollary 3.1, the next corollary is evident.

Corollary 3.5. *Consider $\mathbb{L}$ being an ideal on a Čech closure space $(\Upsilon, \check{c}l)$. The class of all $\mathbb{L}$g̣-Čopen sets constitutes the infra-topology on Υ.*

4 On $\Omega^{\mathbb{L}}$-Sets

Recently, it has been observed that several articles focus on the investigation of g̣-closed (or open) sets in a Čech closure space, exhibiting properties that are more or less analogous to those of closed (or open) sets. Extensive research on g̣-closed (resp. open) sets of a Čech closure spaces is still being continued by different mathematicians. For instance, these concepts $\Omega()$, $\mho()$ have been studied for any set in [6], where the additive condition was relaxed into monotone condition ($\check{c}l(\psi) \subseteq \check{c}l(M)$, when $\psi \subseteq M$) in Čech closure operator definition. Based on this, Boonpok offered the definitions $\Omega()$, $\mho()$ by the following form:

$$\Omega(\psi) = \cap\{M \in G\check{C}O(\Upsilon) : \psi \subseteq M\},$$
$$\mho(\psi) = \cup\{M \in G\check{C}C(\Upsilon) : M \subseteq \psi\}.$$

In the next portion, $\Omega^{\mathbb{L}}$-sets, $\mho^{\mathbb{L}}$-sets ideas will be introduced, along with some of their key characteristics.

Definition 4.1. *Consider $(\Upsilon, \check{c}l, \mathbb{L})$ being an ideal Čech closure space and $\psi \subseteq \Upsilon$. The family $\Omega^{\mathbb{L}}(\psi)$, and $\mho^{\mathbb{L}}(\psi)$ are defined as:*

$$\Omega^{\mathbb{L}}(\psi) = \cap\{M \in \mathbb{L}\underset{\cdot}{g}\check{C}O(\Upsilon) : \psi \subseteq M\},$$
$$\mho^{\mathbb{L}}(\psi) = \cup\{M \in \mathbb{L}\underset{\cdot}{g}\check{C}C(\Upsilon) : M \subseteq \psi\}.$$

Remark 4.1.

- *$\Omega^{\mathbb{L}}(\psi)$ represents the smallest generalized Čech open set relative to the ideal $\mathbb{L}$ that contains ψ.*
- *$\mho^{\mathbb{L}}(\psi)$ represents the greatest generalized Čech closed set relative to the ideal $\mathbb{L}$ that contained in ψ.*
- *The class of all Ω-sets is contained in a class of all $\Omega^{\mathbb{L}}$-sets. If $\mathbb{L} = \{\emptyset\}$, then Definition 4.1, with Definition 3.1 in article [6] are coinciding.*

Theorem 4.1. *Let ψ, M be subsets of an ideal Čech closure space $(\Upsilon, \check{c}l, \mathbb{L})$. The next statements are therefore valid:*

1. $M \subseteq \Omega^{\mathbb{L}}(M)$.
2. $M = \Omega^{\mathbb{L}}(M)$, if $M \in \mathbb{L}\underset{\cdot}{g}\check{C}O(\Upsilon)$.
3. $\Omega^{\mathbb{L}}(\psi) \subseteq \Omega^{\mathbb{L}}(M)$, if $\psi \subseteq M$.
4. $\Omega^{\mathbb{L}}(\psi) \cup \Omega^{\mathbb{L}}(M) \subseteq \Omega^{\mathbb{L}}(\psi \cup M)$.
5. $\Omega^{\mathbb{L}}(\psi \cap M) \subseteq \Omega^{\mathbb{L}}(\psi) \cap \Omega^{\mathbb{L}}(M)$.
6. $\Omega^{\mathbb{L}}(\Omega^{\mathbb{L}}(M)) = \Omega^{\mathbb{L}}(M)$.
7. $\Omega^{\mathbb{L}}(\Upsilon \setminus M) = \Upsilon \setminus (\mho^{\mathbb{L}}(M))$.
8. $\Omega^{\mathbb{L}}(\emptyset) = \emptyset$, $\Omega^{\mathbb{L}}(\Upsilon) = \Upsilon$.

Proof. We will provide a proof for item 6, as the proofs of the remaining points follow directly from Definition 4.1.

6. $\Omega^{\mathbb{L}}(\Omega^{\mathbb{L}}(M)) = \Omega^{\mathbb{L}}(M)$**:**
   From items 1 and 3 of Theorem 4.1, we have
   $$\Omega^{\mathbb{L}}(M) \subseteq \Omega^{\mathbb{L}}(\Omega^{\mathbb{L}}(M)).$$
   To establish the reverse inclusion, let $q \in \Omega^{\mathbb{L}}(\Omega^{\mathbb{L}}(M))$. By definition, this means that for every $\mathbb{L}$g̣-Čopen set U containing q, we have
   $$\Omega^{\mathbb{L}}(M) \subseteq U.$$
   Since it follows from the definition that $M \subseteq \Omega^{\mathbb{L}}(M)$, we conclude that $q \in \Omega^{\mathbb{L}}(M)$. Hence,
   $$\Omega^{\mathbb{L}}(\Omega^{\mathbb{L}}(M)) = \Omega^{\mathbb{L}}(M).$$

□

Remark 4.2. *Based on Theorem 4.1, $(\Upsilon, \Omega^{\mathbb{L}})$ constitutes a convex closure space as described by Stadler [29].*

The next example indicates that the converse of items 1, 3 and 5 of Theorem 4.1 are incorrect.

Example 4.1. *Let $\Upsilon = \{1, 2, 3\}$. Define Čech closure operator as follows:*

$$\check{c}l(\emptyset) = \emptyset, \qquad \check{c}l(\{1\}) = \{1,2\}, \qquad \check{c}l(\{2\}) = \{2\}, \qquad \check{c}l(\{3\}) = \{3\},$$
$$\check{c}l(\{1,2\}) = \check{c}l(\{1,3\}) = \check{c}l(\{2,3\}) = \check{c}l(\Upsilon) = \Upsilon.$$

Suppose that $\mathbb{L} = \{\emptyset, \{2\}\}$ is an ideal on Υ. Then,

- $\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{1,2\}, \{1,3\}\}$.
- $G\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{1,2\}, \{1,3\}, \{1\}\}$.
- $\mathbb{L}\underset{\cdot}{g}\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{1,3\}, \{1,2\}, \{3\}, \{1\}\}$.
- $\Omega(\{1\}) = \{1\}$, $\Omega(\{2\}) = \{1,2\}$, $\Omega(\{3\}) = \{1,3\}$, $\Omega(\{1,2\}) = \{1,2\}$, $\Omega(\{1,3\}) = \{1,3\}$, $\Omega(\{2,3\}) = \Upsilon$.
- $\Omega^{\mathbb{L}}(\{1\}) = \{1\}$, $\Omega^{\mathbb{L}}(\{2\}) = \{1,2\}$, $\Omega^{\mathbb{L}}(\{3\}) = \{3\}$, $\Omega^{\mathbb{L}}(\{1,2\}) = \{1,2\}$, $\Omega^{\mathbb{L}}(\{1,3\}) = \{1,3\}$, $\Omega^{\mathbb{L}}(\{2,3\}) = \Upsilon$.

and

1: *If* $M = \{2,3\}$, *then* $\Omega^{\mathbb{L}}(M) = \Upsilon$. *Hence* $\Omega^{\mathbb{L}}(M) \nsubseteq M$.

3: *Let* $\psi = \{1\}$ *and* $M = \{2,3\}$, *then* $\Omega^{\mathbb{L}}(\psi) = \{1\}$ *and* $\Omega^{\mathbb{L}}(M) = \Upsilon$. *Hence,* $\Omega^{\mathbb{L}}(\psi) \subseteq \Omega^{\mathbb{L}}(M)$, *but* $\psi \nsubseteq M$.

5: *Let* $\psi = \{1,3\}$ *and* $M = \{2,3\}$, *then* $\Omega^{\mathbb{L}}(\psi) = \{1,3\}$, $\Omega^{\mathbb{L}}(M) = \Upsilon$ *and* $\Omega^{\mathbb{L}}(\psi \cap M) = \{3\}$. *Hence,* $\Omega^{\mathbb{L}}(\psi \cap M) \neq \Omega^{\mathbb{L}}(\psi) \cap \Omega^{\mathbb{L}}(M)$.

The next example demonstrates that the converse of item 4 of Theorem 4.1 is false.

Example 4.2. *Continued in Example 3.5.*

Then,

- $\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}\}$.
- $G\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{b\}, \{c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}\}$.
- $\mathbb{L}\underset{\cdot}{g}\check{C}O(\Upsilon) = \{\Upsilon, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b,c\}, \{c,d\}, \{b,d\}, \{a,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$.

Let $\psi = \{a\}$ *and* $M = \{b\}$, *then* $\Omega^{\mathbb{L}}(\psi) = \{a\}$, $\Omega^{\mathbb{L}}(M) = \{b\}$ *and* $\Omega^{\mathbb{L}}(\psi \cup M) = \{a,b,c\}$. *Hence,* $\Omega^{\mathbb{L}}(\psi) \cup \Omega^{\mathbb{L}}(M) \neq \Omega^{\mathbb{L}}(\psi \cup M)$.

As stated in item 7 of Theorem 4.1, the next results hold.

Theorem 4.2. *Let* ψ, M *be two subsets of an ideal Čech closure space* $(\Upsilon, \check{c}l, \mathbb{L})$. *The next statements are therefore valid:*

1. $\mho^{\mathbb{L}}(M) \subseteq M$.
2. $M = \mho^{\mathbb{L}}(M)$, *if* $M \in \mathbb{L}\underset{\cdot}{g}\check{C}C(\Upsilon)$.
3. $\mho^{\mathbb{L}}(\psi) \subseteq \mho^{\mathbb{L}}(M)$, *if* $\psi \subseteq M$.
4. $\mho^{\mathbb{L}}(\psi) \cup \mho^{\mathbb{L}}(M) \subseteq \mho^{\mathbb{L}}(\psi \cup M)$.
5. $\mho^{\mathbb{L}}(\psi \cap M) \subseteq \mho^{\mathbb{L}}(\psi) \cap \mho^{\mathbb{L}}(M)$.
6. $\mho^{\mathbb{L}}(\mho^{\mathbb{L}}(M)) = \mho^{\mathbb{L}}(M)$.
7. $\mho^{\mathbb{L}}(\Upsilon \setminus M) = \Upsilon \setminus (\Omega^{\mathbb{L}}(M))$.
8. $\mho^{\mathbb{L}}(\emptyset) = \emptyset$, $\mho^{\mathbb{L}}(\Upsilon) = \Upsilon$.

Definition 4.2. *A subset* M *of an ideal Čech closure space* $(\Upsilon, \check{c}l, \mathbb{L})$ *is named:*

1. $\Omega^{\mathbb{L}}$*-set, if* $M = \Omega^{\mathbb{L}}(M)$.
2. $\mho^{\mathbb{L}}$*-set, if its complement is* $\Omega^{\mathbb{L}}$*-set i.e* $M = \mho^{\mathbb{L}}(M)$.

The class of all $\Omega^{\mathbb{L}}$ *(resp.* $\mho^{\mathbb{L}}$*) -sets denote by* $\Omega^{\mathbb{L}}(\Upsilon, \check{c}l)$ *(resp.* $\mho^{\mathbb{L}}(\Upsilon, \check{c}l)$*).*

Example 4.3. *Continued in Example 4.2.*

- $\Omega(\Upsilon, \check{c}l) = G\check{C}O(\Upsilon)$.

- $\Omega^{\mathbb{L}}(\Upsilon, \breve{c}l) = \mathbb{L}\underset{\cdot}{g}\breve{C}O(\Upsilon)$.

According to Theorems 4.1 and 4.2, the next corollary is understandable.

Corollary 4.1. *Let Čech closure spaces $(\Upsilon, \breve{c}l, \mathbb{L})$ be an ideal. Then,*

1. $\Omega^{\mathbb{L}}(M)$ *is* $\Omega^{\mathbb{L}}$*-set,* $\forall M \subseteq \Upsilon$.
2. $\mho^{\mathbb{L}}(M)$ *is* $\mho^{\mathbb{L}}$*-set,* $\forall M \subseteq \Upsilon$.
3. $\psi \cap M$ *is* $\Omega^{\mathbb{L}}$*-set, if* ψ, M *are* $\Omega^{\mathbb{L}}$*-sets.*
4. $\psi \cup M$ *is* $\mho^{\mathbb{L}}$*-set, if* ψ, M *are* $\mho^{\mathbb{L}}$*-sets*
5. $\Upsilon, \emptyset$ *are* $\Omega^{\mathbb{L}}$*-sets.*
6. $\Upsilon, \emptyset$ *are* $\mho^{\mathbb{L}}$*-sets.*

Remark 4.3. *Let $(\Upsilon, \breve{c}l, \mathbb{L})$ be an ideal Čech closure space. Hence, the class of $\Omega^{\mathbb{L}}$-sets (resp. $\mho^{\mathbb{L}}$-sets) form an infra topology on Υ.*

Definition 4.3. *A map Φ between two ideal Čech closure spaces $(\Upsilon, \breve{c}l, \mathbb{L})$, $(\Lambda, \breve{c}l^{\star}, \Phi(\mathbb{L}))$ is said to be:*

1. $\Omega^{\mathbb{L}}$*-map if* $\Phi(U) \in \Phi(\mathbb{L})G\breve{C}C(\Lambda)$ *for all* $U \in \Omega^{\mathbb{L}}(\Upsilon, \breve{c}l)$.
2. $\mho^{\mathbb{L}}$*-map if* $\Phi(V) \in \Phi(\mathbb{L})G\breve{C}O(\Lambda)$ *for all* $V \in \mho^{\mathbb{L}}(\Upsilon, \breve{c}l)$.

One can note that the ideas of $\Omega^{\mathbb{L}}$-map and $\mho^{\mathbb{L}}$-map are independent, as demonstrated in the next examples.

Example 4.4. *Let $\Upsilon = \Lambda = \{1, 2, 3\}$, and consider the function $\Phi : (\Upsilon, \breve{c}l, \mathbb{L}) \longrightarrow (\Lambda, \breve{c}l^{\star}, \Phi(\mathbb{L}))$ defined as follows:*

$$\Phi(1) = \Phi(3) = 3, \Phi(2) = 1.$$

A Čech closure operator $\breve{c}l : P(\Upsilon) \longrightarrow P(\Upsilon)$ is assigned as:

$$\breve{c}l(\emptyset) = \emptyset, \qquad \breve{c}l(\{1\}) = \{1\}, \qquad \breve{c}l(\{2\}) = \{2\}, \qquad \textit{and}$$
$$\breve{c}l(\{3\}) = \breve{c}l(\{1,2\}) = \breve{c}l(\{1,3\}) = \breve{c}l(\{2,3\}) = \breve{c}l(\Upsilon) = \Upsilon.$$

Similarly, the Čech closure operator $\breve{c}l^{\star} : P(\Lambda) \longrightarrow P(\Lambda)$ is defined as:

$$\breve{c}l^{\star}(\emptyset) = \emptyset, \qquad \breve{c}l^{\star}(\{1\}) = \{1\}, \qquad \textit{and}$$
$$\breve{c}l^{\star}(\{2\}) = \breve{c}l^{\star}(\{3\}) = \breve{c}l^{\star}(\{1,2\}) = \breve{c}l^{\star}(\{1,3\}) = \breve{c}l^{\star}(\{2,3\}) = \breve{c}l^{\star}\Lambda = \Lambda.$$

If $\mathbb{L} = \{\emptyset, \{3\}\}$, then the function Φ is an $\Omega^{\mathbb{L}}$-map but not an $\mho^{\mathbb{L}}$-map. This follows from the fact that $\{1, 2\} \in \mho^{\mathbb{L}}(\Upsilon, \breve{c}l)$, but its image under Φ is $\Phi(\{1, 2\}) = \{1, 3\}$, which is not an element of $\Phi(\mathbb{L})G\breve{C}O(\Lambda)$ in the structure $(\Lambda, \breve{c}l^{\star}, \Phi(\mathbb{L}))$.

Example 4.5. *Let $\Upsilon = \Lambda = \{1, 2, 3\}$, and consider the function $\Phi : (\Upsilon, \breve{c}l, \mathbb{L}) \longrightarrow (\Lambda, \breve{c}l^{\star}, \Phi(\mathbb{L}))$ defined by:*

$$\Phi(1) = 3, \quad \textit{and} \quad \Phi(2) = \Phi(3) = 2.$$

A closure operator $\check{cl} : P(\Upsilon) \longrightarrow P(\Upsilon)$ *is assigned as follows:*

$$\check{cl}(\emptyset) = \emptyset, \qquad \check{cl}(\{1\}) = \{1\}, \qquad \check{cl}(\{2\}) = \{2\}, \qquad \text{and}$$
$$\check{cl}(\{3\}) = \check{cl}(\{1,2\}) = \check{cl}(\{1,3\}) = \check{cl}(\{2,3\}) = \check{cl}\Upsilon = \Upsilon.$$

Similarly, the closure operator $\check{cl}^\star : P(\Lambda) \longrightarrow P(\Lambda)$ *is defined by:*

$$\check{cl}^\star(\emptyset) = \emptyset, \qquad \check{cl}^\star(\{1\}) = \{1\}, \qquad \text{and}$$
$$\check{cl}^\star(\{2\}) = \check{cl}^\star(\{3\}) = \check{cl}^\star(\{1,2\}) = \check{cl}^\star(\{1,3\}) = \check{cl}^\star(\{2,3\}) = \check{cl}^\star\Lambda = \Lambda.$$

Let $\mathbb{L} = \{\emptyset, \{1\}\}$*. Then, the function* Φ *is an* $\mho^{\mathbb{L}}$*-map but not an* $\Omega^{\mathbb{L}}$*-map. This follows from the fact that* $\{2,3\} \in \Omega^{\mathbb{L}}(\Upsilon, \check{cl}^\star)$*, but its image under* Φ*, given by* $\Phi(\{2,3\}) = \{2\}$*, is not a subset of* $\Phi(\mathbb{L})G\check{C}C(\Lambda)$ *in the structure* $(\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$*.*

Theorem 4.3. *Let* $\Phi : (\Upsilon, \check{cl}, \mathbb{L}) \longrightarrow (\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$ *be an* $\Omega^{\mathbb{L}}$*-map between two ideal Čech closure spaces* $(\Upsilon, \check{cl}, \mathbb{L})$*,* $(\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$*. Then, for any subset* $\psi \subseteq \Lambda$ *and any* $F \in \mho^{\mathbb{L}}(\Upsilon, \check{cl})$ *with* $\Phi^{-1}(\psi) \subseteq F$*, there exists a set* $E \in \Phi(\mathbb{L})G\check{C}O(\Lambda)$ *satisfying* $\psi \subseteq E$ *and* $\Phi^{-1}(E) \subseteq F$*.*

Proof. Let $\psi \subseteq \Lambda$ and $F \in \mho^{\mathbb{L}}(\Upsilon, \check{cl})$ with $\Phi^{-1}(\psi) \subseteq F$. Define the set:

$$E = \Lambda \setminus \Phi(\Upsilon \setminus F).$$

Since Φ is an $\Omega^{\mathbb{L}}$-map, it follows that $\Phi(\Upsilon \setminus F) \in \Phi(\mathbb{L})G\check{C}C(\Lambda)$. Therefore, $E \in \Phi(\mathbb{L})G\check{C}O(\Lambda)$.

Given that $\Phi^{-1}(\psi) \subseteq F$, we obtain:

$$\Phi(\Upsilon \setminus F) \subseteq \Lambda \setminus \psi.$$

Consequently,

$$\psi \subseteq \Lambda \setminus \Phi(\Upsilon \setminus F) = E.$$

Furthermore, we establish that,

$$\Phi^{-1}(E) = \Upsilon \setminus \Phi^{-1}(\Phi(\Upsilon \setminus F)) \subseteq \Upsilon \setminus (\Upsilon \setminus F) = F.$$

□

As stated in Theorem 4.3, the proof of the next theorem is obvious.

Theorem 4.4. *Let* $\Phi : (\Upsilon, \check{cl}, \mathbb{L}) \longrightarrow (\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$ *be a* $\mho^{\mathbb{L}}$*-map between two ideal Čech closure spaces* $(\Upsilon, \check{cl}, \mathbb{L})$*,* $(\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$*. Then, for any subset* $M \subseteq \Lambda$ *such that* $M \in \Omega^{\mathbb{L}}(\Upsilon, \check{cl})$ *and* $\Phi^{-1}(M) \subseteq M$*, there exists a set* $S \in \Phi(\mathbb{L})G\check{C}C(\Lambda)$ *satisfying* $M \subseteq S$ *and* $\Phi^{-1}(S) \subseteq M$*.*

Definition 4.4. *A map* Φ *between two ideal Čech closure spaces* $(\Upsilon, \check{cl}, \mathbb{L})$*,* $(\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$ *is called* $\mho^\bullet$*-closed if* $\Phi(S)$ *is* $\mho^{\Phi(\mathbb{L})}$*-set in* $(\Lambda, \check{cl}^\star, \Phi(\mathbb{L}))$ *for every Čclosed subset* S *of* $(\Upsilon, \check{cl}, \mathbb{L})$*.*

Theorem 4.5. *Let* $(\Upsilon, \check{cl}, \mathbb{L})$*,* $(\Lambda, \check{cl}^\star, \dot{f}(\mathbb{L}))$*, and* $(\Xi, \check{cl}^{\star\star}, (\dot{g} \circ \dot{f})(\mathbb{L}))$ *be ideal Čech closure spaces. If* $\dot{f} : (\Upsilon, \check{cl}, \mathbb{L}) \longrightarrow (\Lambda, \check{cl}^\star, \dot{f}(\mathbb{L}))$*,* $\dot{g} : (\Lambda, \check{cl}^\star, \dot{f}(\mathbb{L})) \longrightarrow (\Xi, \check{cl}^{\star\star}, (\dot{g} \circ \dot{f})(\mathbb{L}))$ *are mappings such that* $\dot{g} \circ \dot{f} : (\Upsilon, \check{cl}, \mathbb{L}) \longrightarrow (\Xi, \check{cl}^{\star\star}, (\dot{g} \circ \dot{f})(\mathbb{L}))$ *is an* $\mho^\bullet$*-closed map. Then* $\dot{g}$ *is also an* $\mho^\bullet$*-closed map, provided that* $\dot{f}$ *is Čech-continuous and surjective.*

Proof. Let S be a Čclosed subset of $(\Lambda, \breve{c}l^{\star}, \dot{f}(\mathbb{L}))$. Since $\dot{f}$ is Čech-continuous, it follows that $\dot{f}^{-1}(S)$ is a Čclosed subset of $(\Upsilon, \breve{c}l, \mathbb{L})$. Given that $(\dot{g} \circ \dot{f})$ is an $\mho^{\bullet}$-closed map and $\dot{f}$ is surjective, we conclude that,

$$\dot{g}(S) = (\dot{g} \circ \dot{f})(\dot{f}^{-1}(S)).$$

Since $\dot{f}^{-1}(S)$ is Čclosed in $(\Upsilon, \breve{c}l, \mathbb{L})$, the $\mho^{\bullet}$-closedness of $\dot{g} \circ \dot{f}$ ensures that $\dot{g}(S)$ is an $\mho^{(\dot{g}\circ\dot{f})(\mathbb{L})}$-set in $(\Xi, \breve{c}l^{\star\star}, (\dot{g} \circ \dot{f})(\mathbb{L}))$. □

5 Conclusion and Future Work

In the scope of Čech closure spaces, this study introduces and systematically develops the concepts of g-closed and g-open sets, enriched by the presence of ideals. These newly proposed classes, namely the $\mathbb{L}$ġ-Čclosed and $\mathbb{L}$ġ-Čopen sets, extend both classical and generalized closedness notions by incorporating the structure of an ideal $\mathbb{L}$, offering a more flexible and robust topological framework.

Several foundational properties of these sets were rigorously established, including their behavior under set-theoretic operations and their relationships with the standard Čech closure operator. The theoretical development was complemented by the introduction of the operators

$$\Omega^{\mathbb{L}}(\psi) = \bigcap\{M \in \mathbb{L}\dot{g}\breve{C}O(\Upsilon) \mid \psi \subseteq M\}, \quad \mho^{\mathbb{L}}(\psi) = \bigcup\{M \in \mathbb{L}\dot{g}\breve{C}C(\Upsilon) \mid M \subseteq \psi\},$$

which were shown to act as closure and interior operators, respectively, in the context of ideal Čech closure spaces. Sets fixed under $\Omega^{\mathbb{L}}$ and $\mho^{\mathbb{L}}$ were identified precisely as the $\mathbb{L}$ġ-Čclosed and $\mathbb{L}$ġ-Čopen sets, revealing a rich algebraic structure. Importantly, the paper also presented a computational algorithm for identifying $\mathbb{L}$ġ-Čclosed sets, enhancing the practical utility of the theory and facilitating its application in real-world problems. Moreover, it was demonstrated that many results from existing literature emerge as special cases within the more general framework developed in this work. The behavior of these generalized sets under mappings was explored, revealing preservation results for preimages and images under Čech-continuous and open functions. Examples were provided to illustrate that converses of such results do not generally hold, highlighting the novelty and generality of the introduced notions.

The findings of this study have potential applications across various domains, including fuzzy topological applications of [3], rough set theory and decision-making problems. These applications are evident in the context of rough set generalizations [13, 15], decision-making processes for diagnosing heart failure [30, 31], and other areas within medical research [2, 25]. Therefore, our future work will be concentrated on these domains, where the ideal-theoretic generalizations of Čech closure can contribute to more refined models of approximation and uncertainty. We hope that the findings presented in this manuscript will inspire further research into ideal Čech closure spaces. Potential directions include the study of separation axioms in this setting, connections with fuzzy and rough set theory, and the expansion of these ideas to other generalized closure frameworks. Ultimately, we aim to foster the development of a more comprehensive and applicable topological structure that aligns with both theoretical insights and practical demands.

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